Logarithms

Jelani Nelson
UC Berkeley

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- $10^3 = 1000$
  $\log_{10}(1000) = 3$
  "the logarithm base 10 of 1000 is 3"

- $2^5 = 32$
  $\log_2(32) = 5$
  "the logarithm base 2 of 32 is 5"

$\log_{10}(1) = 0$

$\log_{10}(10) = 1$

$\log_{10}(100) = 2$

$\log_{10}(1{,}000) = 3$

$\log_{10}(10{,}000) = 4$

$\log_{10}(100{,}000) = 5$

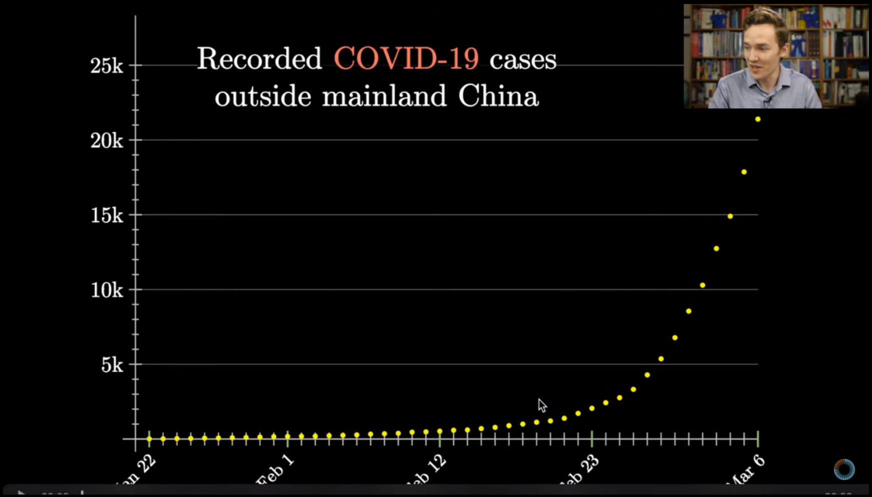
$\log_{10}(1{,}000{,}000) = 6$

$\log_{10}(10{,}000{,}000) = 7$

$\log_{10}(100{,}000{,}000) = 8$

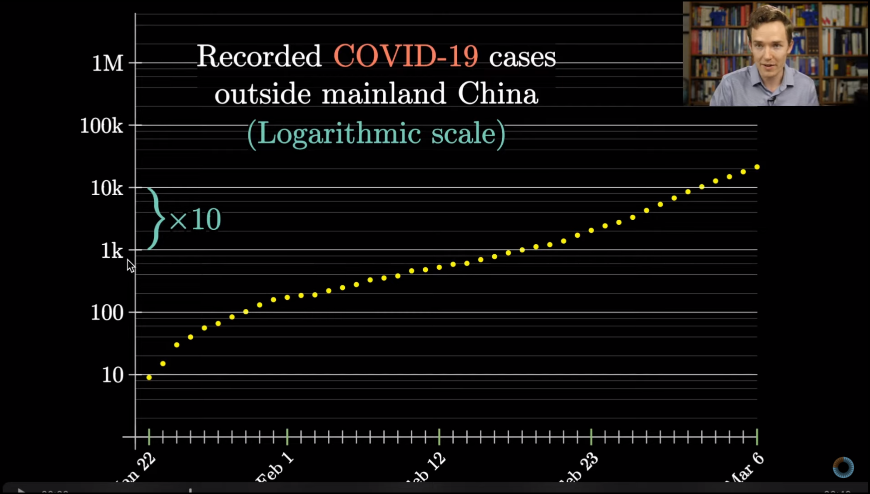
$\log_{10}(1{,}000{,}000{,}000) = 9$

When the base is 10, and the argument x is a power of 10, $\log_{10}(x)$ counts number of zeroes.



$f(x)$ = number of recorded COVID-19 cases on day x

source: 3Blue1Brown (https://www.youtube.com/watch?v=cEvgcoyZvB4)



$$g(x) = \log_{10}(\text{number of recorded COVID-19 cases on day } x)$$

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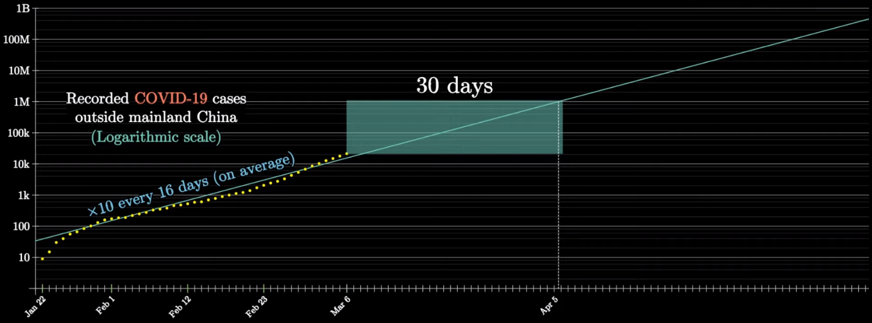

On what day do we expect to cross 1 million recorded cases?

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Starting from day 1, how long before 1 million cases?

$$1.1548^x = 1{,}000{,}000 \text{ means } x = \log_{1.1548}(1{,}000{,}000) \text{ days}$$

(96 days)

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Indeed, $a^x = b, a^y = c$ means $a^{x+y} = b \cdot c$
(x is $\log_a(b)$ and y is $\log_a(c)$)

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Indeed, if $a^x = b$ then $b^n = (a^x)^n = a^{n \cdot x}$
(x is $\log_a(b)$)

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